

# $RP^{[d]}$ IS AN EQUIVALENCE RELATION AN ENVELOPING SEMIGROUP PROOF

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ABSTRACT. We present a purely enveloping semigroup proof of a theorem of Shao and Ye which asserts that for an abelian group  $T$ , a minimal flow  $(X, T)$  and any integer  $d \geq 1$ , the regional proximal relation of order  $d$  is an equivalence relation.

Let  $T$  be a countable abelian group and let  $(X, T)$  be a minimal flow; i.e.  $X$  is a compact Hausdorff space and  $T$  acts on it as a group of homeomorphisms in such a way that for each  $x \in X$  its  $T$ -orbit,  $Tx = \{tx : t \in T\}$ , is dense in  $X$ . Following [6] and [9] we introduce the following notations (generalizing from the case  $T = \mathbb{Z}$  to the case of a general  $T$  action). For an integer  $d \geq 1$  let  $X^{[d]} = X^{2^d}$ . We index the coordinates of an element  $x \in X^{[d]}$  by subsets  $\epsilon \subset \{1, \dots, d\}$ . Thus  $x = (x_\epsilon : \epsilon \subset \{1, \dots, d\})$ , where for each  $\epsilon$ ,  $x_\epsilon \in X_\epsilon = X$ . E.g. for  $d = 2$  we have  $x = (x_\emptyset, x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}})$ .

Let  $\pi_* : X^{[d]} \rightarrow X^{2^{d-1}}$  denote the projection onto the last  $2^{d-1}$ -coordinates; i.e., the map which forgets the  $\emptyset$ -coordinate. Let  $X_*^{[d]} = \pi_*(X^{[d]}) = X^{2^{d-1}} = \prod \{X_\epsilon : \epsilon \neq \emptyset\}$  and for  $x \in X^{[d]}$  let  $x_* = \pi_*(x) \in X_*^{[d]}$  denote its projection; i.e.  $x_*$  is obtained by omitting the  $\emptyset$ -coordinate of  $x$ . For each  $\epsilon \subset \{1, \dots, d\}$  we denote by  $\pi_\epsilon$  the projection map from  $X^{[d]}$  onto  $X_\epsilon = X$ . For a point  $x \in X$  we let  $x^{[d]} \in X^{[d]}$  and  $x_*^{[d]} \in X_*^{[d]}$  be the *diagonal points* all of whose coordinates are  $x$ .  $\Delta^{[d]} = \{x^{[d]} : x \in X\}$  is the *diagonal* of  $X^{[d]}$  and  $\Delta_*^{[d]} = \{x_*^{[d]} : x \in X\}$  the *diagonal* of  $X_*^{[d]}$ . Another convenient representation of  $X^{[d]}$  is as a product space  $X^{[d]} = X^{[d-1]} \times X^{[d-1]}$  (with  $X^{[0]} = X$ ). When using this decomposition we write  $x = (x', x'')$ . More explicitly, for  $\epsilon \subset \{1, \dots, d-1\}$  let  $\epsilon d = \epsilon \cup \{d\}$ , and define the identification  $X^{[d-1]} \times X^{[d-1]} \rightarrow X^{[d]}$  by  $(x', x'') \mapsto x$  with  $x_\epsilon = x'_\epsilon$  and  $x_{\epsilon d} = x''_\epsilon$ . We will refer to  $x'$  and  $x''$  as the first and second  $2^{d-1}$  coordinates, respectively.

We next define two group actions on  $X^{[d]}$ , the *face group action*  $\mathcal{F}_d$  and the *total group action*  $\mathcal{G}_d$ . These actions are representations of  $T^d = T \times T \times \dots \times T$  ( $d$  times) and  $T^{d+1}$ , respectively, as subgroups of  $\text{Homeo}(X^{[d]})$ . For the  $\mathcal{F}_d$  action,  $\mathcal{F}_d \times X^{[d]} \rightarrow X^{[d]}$ ,

$$((t_1, \dots, t_d), (x_\epsilon : \epsilon \subset \{1, \dots, d\})) \mapsto (t_\epsilon x_\epsilon : \epsilon \subset \{1, \dots, d\}),$$

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where  $t_\epsilon x_\epsilon = t_{n_k} \cdots t_{n_1} x_\epsilon$ , if  $\epsilon = \{n_1, \dots, n_k\}$  and  $t_\emptyset x_\emptyset = x_\emptyset$ . We can then represent the homeomorphism  $\tau \in \mathcal{F}_d$  which corresponds to  $(t_1, \dots, t_d) \in T^d$  as

$$\tau = \tau_{(t_1, \dots, t_d)}^{[d]} = (t_\epsilon : \epsilon \subset \{1, \dots, d\}).$$

We will also consider the restriction of the  $\mathcal{F}_d$  action to  $X_*^{[d]}$  which is defined by omitting the  $\emptyset$  coordinate. Note that under the action of  $\mathcal{F}_d$  on  $X^{[d]}$  the  $\emptyset$ -coordinate is left fixed.

For example, if we consider a minimal cascade  $(X, f)$ , taking  $T = \mathbb{Z} = \{f^n : n \in \mathbb{Z}\}$ ,  $d = 3$  and  $\tau = \tau_{(2,5,11)}^{[3]} \in \mathcal{F}_3 \cong \mathbb{Z}^3$ , we have:

$$\tau(x) = (x_\emptyset, f^2 x_{\{1\}}, f^5 x_{\{2\}}, f^{2+5} x_{\{1,2\}}, f^{11} x_{\{3\}}, f^{2+11} x_{\{1,3\}}, f^{2+5+11} x_{\{1,2,3\}}),$$

and

$$\tau(x_*) = (f^2 x_{\{1\}}, f^5 x_{\{2\}}, f^{2+5} x_{\{1,2\}}, f^{11} x_{\{3\}}, f^{2+11} x_{\{1,3\}}, f^{2+5+11} x_{\{1,2,3\}}).$$

Note that the fact that the  $\mathcal{F}_d$  action is well defined depends on the commutativity of the group  $T$ .

The action of  $T^{d+1}$  on  $X^{[d]}$ , denoted by  $\mathcal{G}_d$ , is the action generated by the face group action  $\mathcal{F}_d$  and the *diagonal  $\theta$ -action* of  $T$ ,  $T \times X^{[d]} \rightarrow X^{[d]}$ , defined by

$$(t, x) \mapsto \theta_t^{[d]} x = (tx_\epsilon : \epsilon \subset \{1, \dots, d\}).$$

Thus for the  $\mathcal{G}_d$  action  $\mathcal{G}_d \times X^{[d]} \rightarrow X^{[d]}$ ,

$$((t_1, \dots, t_d, t_{d+1}), (x_\epsilon : \epsilon \subset \{1, \dots, d\})) \mapsto (t_{d+1} t_\epsilon x_\epsilon : \epsilon \subset \{1, \dots, d\}),$$

where  $t_\epsilon x_\epsilon = t_{n_k} \cdots t_{n_1} x_\epsilon$ , if  $\epsilon = \{n_1, \dots, n_k\}$  and  $t_\emptyset x_\emptyset = x_\emptyset$ . In other words, the  $\mathcal{G}_d$  action on  $X^{[d]}$  is given by the representation:

$$T^{d+1} \rightarrow \text{Homeo}(X^{[d]}), \quad (t_1, \dots, t_d, t_{d+1}) \mapsto \theta_{t_{d+1}}^{[d]} \tau_{(t_1, \dots, t_d)}^{[d]}.$$

Notice that

$$(1) \quad \tau_{(t_1, \dots, t_d)}^{[d]}(x', x'') = (\tau_{(t_1, \dots, t_{d-1})}^{[d-1]} x', \theta_{t_d}^{[d-1]} \tau_{(t_1, \dots, t_{d-1})}^{[d-1]} x'').$$

In their paper [9] Shao and Ye prove that  $RP^{[d]}$ , the generalized regionally proximal relation of order  $d$ , is always an equivalence relation for a minimal cascade  $(X, T)$ . Their proof is based on the detailed analysis of the  $\mathcal{G}_d$  action provided by Host Kra and Mass in [6], where the authors treated the distal case. The main tool used by Shao and Ye is a theorem which asserts that, for each  $x \in X$ , the action of the face group  $\mathcal{F}_d$  on the orbit closure  $\text{cls } \mathcal{F}_d x_*^{[d]}$  is minimal. Their proof of this latter theorem was based on the general structure theory of minimal flows due to Ellis-Glasner-Shapiro [3], McMahon [8] and Veech [10]. Now, it turns out that there is a direct enveloping semigroup proof of this theorem which is very similar to the proof by Ellis and Glasner given in [7, page 46]. The possibility of applying the Ellis-Glasner proof as a shortcut to Shao and Ye's proof was also discovered by Ethan Akin. In the next section I present this short proof, established for a general commutative group. For the interested reader I will, in a subsequent section, reproduce the beautiful proof of Shao and Ye of the fact that for each  $d \geq 1$ ,  $RP^{[d]}$  is an equivalence relation.

Let us note that all the results of this work extend to the case where  $T$  is a commutative separable topological group which acts continuously and minimally on

a compact metric space  $X$ . In fact, both minimality and the regionally proximal relations are the same for the group and for a countable dense subgroup. Moreover, most of the results (like Theorem 1.1, parts (1) - (3), as well as Theorem 2.5) hold for actions of  $T$  on a general compact space (not necessarily metrizable). I wish to thank Ethan Akin whose suggestions led to improvements of a first draft of this work.

### 1. THE MINIMALITY OF THE FACE ACTION ON $Q_{x*}^{[d]}$

Let  $(X, T)$  be a minimal flow with  $T$  abelian. Let

$$Q^{[d]} = \text{cls} \{gx^{[d]} : x \in X, g \in \mathcal{G}_d\} = \overline{\mathcal{G}_d \Delta^{[d]}} = \overline{\mathcal{F}_d \Delta^{[d]}},$$

$$Q_x^{[d]} = Q^{[d]} \cap (\{x\} \times X^{2^d-1}), \text{ and } Q_{x*}^{[d]} = \pi_*(Q_x^{[d]}).$$

For each  $x \in X$  let  $Y_x^{[d]} = \overline{\mathcal{F}_d(x^{[d]})} \subset Q_x^{[d]}$  be the orbit closure of  $x^{[d]}$  under  $\mathcal{F}_d$ . Finally, let  $Y_{x*}^{[d]} = \pi_*(Y_x^{[d]})$ .

- 1.1. **Theorem** (Shao and Ye). 1. *The flow  $(Q^{[d]}, \mathcal{G}_d)$  is minimal.*  
 2. *For each  $x \in X$ , the flows  $(Y_x^{[d]}, \mathcal{F}_d)$ , and hence also  $(Y_{x*}^{[d]}, \mathcal{F}_d)$ , are minimal.*  
 3. *For each  $x \in X$  the set  $Y_x^{[d]}$  is the unique minimal subflow of the  $\mathcal{F}_d$ -flow  $(Q_x^{[d]}, \mathcal{F}_d)$ . Hence also  $Y_{x*}^{[d]}$  is the unique minimal subflow of the  $\mathcal{F}_d$ -flow  $(Q_{x*}^{[d]}, \mathcal{F}_d)$ .*  
 4. <sup>1</sup> *For a dense  $G_\delta$  subset  $X_0 \subset X$  we have  $Y_x^{[d]} = Q_x^{[d]}$ .*

*Proof.* 1. Let us denote  $N := Q^{[d]}$  and  $\mathcal{T} := \mathcal{G}_d$ . Let  $E = E(N, \mathcal{T})$  be the enveloping semigroup of  $(N, \mathcal{T})$ . Let  $\pi_\epsilon : N \rightarrow X_\epsilon = X$  be the projection of  $N$  on the  $\epsilon$  coordinate, where  $\epsilon \in \{1, \dots, d\}$ . We consider the action of the group  $\mathcal{T}$  on the  $\epsilon$  coordinate via the projection  $\pi_\epsilon$ , that is, for  $\epsilon \in \{1, \dots, d\}$ ,  $(t_1, \dots, t_d, t_{d+1}) \in T^{d+1}$  and  $x \in X_\epsilon = X$ ,

$$\mathcal{T} \times X_\epsilon \rightarrow X_\epsilon, \quad (\theta_{t_{d+1}}^{[d]} \tau_{(t_1, \dots, t_d)}^{[d]}, x) \mapsto t_{d+1} t_\epsilon x.$$

With respect to this action of  $\mathcal{T}$  on  $X_\epsilon = X$  the map  $\pi_\epsilon : (N, \mathcal{T}) \rightarrow (X_\epsilon, \mathcal{T})$  is a flow homomorphism. Let  $\pi_\epsilon^\bullet : E(N, \mathcal{T}) \rightarrow E(X_\epsilon, \mathcal{T})$  be the corresponding homomorphism of enveloping semigroups. Notice that for the action of  $\mathcal{T}$  on  $X_\epsilon$ ,  $E(X_\epsilon, \mathcal{T}) = E(X, T)$  as subsets of  $X^X$  (as  $t_{d+1} t_\epsilon \in T$ ).

Let now  $u \in E(X, T)$  be any minimal idempotent. Then  $\tilde{u} = (u, u, \dots, u) \in E(N, \mathcal{T})$ . Choose  $v$  a minimal idempotent in the closed left ideal  $E(N, \mathcal{T})\tilde{u}$ . Then  $v\tilde{u} = v$ . We want to show that  $\tilde{u}v = \tilde{u}$ . Set, for  $\epsilon \in \{1, \dots, d\}$ ,  $v_\epsilon = \pi_\epsilon^\bullet v$ . Note that, as an element of  $E(N, \mathcal{T})$  is determined by its projections, it suffices to show that for each  $\epsilon$ ,  $uv_\epsilon = u$ . Since for each  $\epsilon$  the map  $\pi_\epsilon^\bullet$  is a semigroup homomorphism, we have that  $v_\epsilon u = v_\epsilon$  as  $v\tilde{u} = v$ . In particular we deduce that  $v_\epsilon$  is an idempotent belonging to the minimal left ideal  $E(X_\epsilon, T)u = E(X, T)u$  which contains  $u$ . This implies (see [7, Exercise 1.23.2.(b)]) that

$$uv_\epsilon = u,$$

and it follows that indeed  $\tilde{u}v = \tilde{u}$ . Thus,  $\tilde{u}$  is an element of the minimal left ideal  $E(N, \mathcal{T})v$  which contains  $v$ , and therefore  $\tilde{u}$  is a minimal idempotent of  $E(N, \mathcal{T})$ .

Now let  $x \in X$  and let  $u$  be a minimal idempotent in  $E(X, T)$  with  $ux = x$  (since  $(X, T)$  is minimal there always exists such an idempotent). By the above argument,  $\tilde{u}$

<sup>1</sup>This seems to be a new observation.

is also a minimal idempotent of  $(N, \mathcal{T})$  which implies that  $N = Q^{[d]}$ , the orbit closure of  $x^{[d]} = \tilde{u}x^{[d]}$ , is  $\mathcal{T}$  minimal (see [7, Exercise 1.26.2]).

2. Given  $x \in X$  we now let  $N := Q_{x*}^{[d]}$  and  $\mathcal{T} := \mathcal{F}_d$ . The proof of the minimality of the flow  $(Q_{x*}^{[d]}, \mathcal{F}_d)$  is almost verbatim the same, except that here the claim that for  $u$  a minimal idempotent in  $E(X, T)$ , the map  $\tilde{u} = (u, u, \dots, u)$  ( $2^d - 1$  times) is in  $E(Q_{x*}^{[d]}, \mathcal{F}_d)$ , is not that evident. However, as  $u$  is an idempotent this fact follows from the following lemma (with  $p_1 = \dots = p_d = u$ ).

**1.2. Lemma.** *Let  $p_1, \dots, p_d \in E(X, T)$  and for  $\epsilon = \{n_1, \dots, n_k\} \subset \{1, \dots, d\}$ , with  $n_1 < \dots < n_k$ , let  $q_\epsilon = p_{n_k} \dots p_{n_1}$ . Then the map  $(q_\epsilon : \epsilon \subset \{1, \dots, d\}, \epsilon \neq \emptyset)$  is an element of  $E(Q_{x*}^{[d]}, \mathcal{F}_d)$ .*

*Proof.* By induction on  $d$ , using the identity (1), or more specifically

$$\tau_{(e, \dots, e, t_d)}^{[d]}(x', x'') = (x', \theta_{t_d}^{[d-1]}x''),$$

and the fact that right multiplication in  $E(X, T)$  is continuous.  $\square$

3. We first reproduce the ingenious “useful lemma” from [9].

**1.3. Lemma.** *If  $(x^{[d-1]}, w) \in Q^{[d]}$  for some  $x \in X$  and  $w \in X^{[d-1]}$  and  $(x^{[d-1]}, w)$  is an  $\mathcal{F}_d$ -minimal point, then  $(x^{[d-1]}, w) \in Y_x^{[d]}$ .*

*Proof.* Since  $(x^{[d-1]}, w) \in Q^{[d]}$  it follows that  $(x^{[d-1]}, w)$  is in the  $\mathcal{G}_d$ -orbit closure of  $x^{[d]}$ , i.e. there is a sequence  $\{(t_{1k}, \dots, t_{dk}, t_{d+1k})\}_{k \in \mathbb{N}} \subset T^{d+1}$  such that

$$\theta_{t_{d+1k}}^{[d]} \tau_{(t_{1k}, \dots, t_{dk})}^{[d]}(x^{[d]}) \rightarrow (x^{[d-1]}, w).$$

Now

$$\begin{aligned} & \theta_{t_{d+1k}}^{[d]} \tau_{(t_{1k}, \dots, t_{dk})}^{[d]}(x^{[d]}) = \\ & \theta_{t_{d+1k}}^{[d]} (\text{id}^{[d-1]} \times \theta_{t_{dk}}^{[d-1]})(\tau_{(t_{1k}, \dots, t_{d-1k})}^{[d-1]}(x^{[d-1]}), \tau_{(t_{1k}, \dots, t_{d-1k})}^{[d-1]}(x^{[d-1]})) = \\ & (\text{id}^{[d-1]} \times \theta_{t_{dk}}^{[d-1]})(\theta_{t_{d+1k}}^{[d-1]} \tau_{(t_{1k}, \dots, t_{d-1k})}^{[d-1]}(x^{[d-1]}), \theta_{t_{d+1k}}^{[d-1]} \tau_{(t_{1k}, \dots, t_{d-1k})}^{[d-1]}(x^{[d-1]})), \end{aligned}$$

and letting  $a_k := \theta_{t_{d+1k}}^{[d-1]} \tau_{(t_{1k}, \dots, t_{d-1k})}^{[d-1]}(x^{[d-1]})$ , we have:

$$(2) \quad (\text{id}^{[d-1]} \times \theta_{t_{dk}}^{[d-1]})(a_k, a_k) \rightarrow (x^{[d-1]}, w).$$

Let

$$\begin{aligned} \pi_1 : (X^{[d]}, \mathcal{F}_d) &\rightarrow (X^{[d-1]}, \mathcal{F}_d), & (x', x'') &\mapsto x', \\ \pi_2 : (X^{[d]}, \mathcal{F}_d) &\rightarrow (X^{[d-1]}, \mathcal{F}_d), & (x', x'') &\mapsto x'', \end{aligned}$$

be the projections to the first and last  $2^{d-1}$  coordinates respectively. For  $\pi_1$  we consider the action of the group  $\mathcal{F}_d$  on  $X^{[d-1]}$  via the representation

$$\tau_{(t_1, \dots, t_d)}^{[d]} \mapsto \tau_{(t_1, \dots, t_{d-1})}^{[d-1]},$$

and for  $\pi_2$  the action is via the representation

$$\tau_{(t_1, \dots, t_d)}^{[d]} \mapsto \theta_{t_d}^{[d-1]} \tau_{(t_1, \dots, t_{d-1})}^{[d-1]}.$$

Denote the corresponding semigroup homomorphisms of enveloping semigroups by

$$\pi_i^\bullet : E(X^{[d]}, \mathcal{F}_d) \rightarrow E(X^{[d-1]}, \mathcal{F}_d), \quad i = 1, 2.$$

Notice that for these actions of  $\mathcal{F}_d$  on  $X^{[d-1]}$ , as subsets of  $X^{[d]X^{[d]}}$ ,

$$\pi_1^\bullet(E(X^{[d]}, \mathcal{F}_d)) = E(X^{[d-1]}, \mathcal{F}_{d-1}) \quad \text{and} \quad \pi_2^\bullet(E(X^{[d]}, \mathcal{F}_d)) = E(X^{[d-1]}, \mathcal{G}_{d-1}).$$

Thus for  $p \in E(X^{[d]}, \mathcal{F}_d)$  and  $x \in X^{[d]}$ , we have:

$$px = p(x', x'') = (\pi_1^\bullet(p)x', \pi_2^\bullet(p)x'').$$

Now fix a minimal left ideal  $L$  of  $E(X^{[d]}, \mathcal{F}_d)$ . By (2)  $a_k \rightarrow x^{[d-1]}$  and, since  $(Q^{[d-1]}, \mathcal{G}_{d-1})$  is minimal, there exist  $p_k \in L$  such that  $a_k = \pi_2^\bullet(p_k)x^{[d-1]}$ . Without loss of generality we assume that  $p_k \rightarrow p \in L$ . Then

$$\pi_2^\bullet(p_k)x^{[d-1]} = a_k \rightarrow x^{[d-1]} \quad \text{and} \quad \pi_2^\bullet(p_k)x^{[d-1]} \rightarrow \pi_2^\bullet(p)x^{[d-1]}.$$

Hence

$$(3) \quad \pi_2^\bullet(p)x^{[d-1]} = x^{[d-1]}.$$

Since  $L$  is a minimal left ideal and  $p \in L$  there exists a minimal idempotent  $v \in J(L)$  such that  $vp = p$ . Then

$$\pi_2^\bullet(v)x^{[d-1]} = \pi_2^\bullet(v)\pi_2^\bullet(p)x^{[d-1]} = \pi_2^\bullet(vp)x^{[d-1]} = \pi_2^\bullet(p)x^{[d-1]} = x^{[d-1]}.$$

Let

$$F = \mathfrak{G}(\overline{\mathcal{F}_{d-1}(x^{[d-1]})}, x^{[d-1]}) = \{\alpha \in vL : \pi_2^\bullet(\alpha)x^{[d-1]} = x^{[d-1]}\}$$

be the Ellis group of the pointed flow  $(\overline{\mathcal{F}_{d-1}(x^{[d-1]})}, x^{[d-1]})$ . Then  $F$  is a subgroup of the group  $vL$ . By (3), we have  $p \in F$  and since  $F$  is a group, we have  $pFx^{[d]} = Fx^{[d]} \subset \pi_2^{-1}(x^{[d]})$ . Since  $vx^{[d]} \in Fx^{[d]} = pFx^{[d]}$ , there is some  $x_0 \in Fx^{[d]}$  such that  $vx^{[d]} = px_0$ . Set  $x_k = p_kx_0$ , then

$$(4) \quad x_k = p_kx_0 \rightarrow px_0 = vx^{[d]} = (\pi_1^\bullet(v)x^{[d-1]}, x^{[d-1]}).$$

As  $x_0 \in Fx^{[d]}$ , it follows that  $\pi_2(x_0) = x^{[d-1]}$ , hence

$$\pi_2(x_k) = \pi_2(p_kx_0) = \pi_2^\bullet(p_k)\pi_2(x_0) = \pi_2^\bullet(p_k)x^{[d-1]} = a_k \rightarrow x^{[d-1]}.$$

Let  $x_k = (b_k, a_k) \in \overline{\mathcal{F}_d(x^{[d]})}$ ; then, by (4),  $\lim b_k = \pi_1^\bullet(v)x^{[d-1]}$ . By (2),  $\theta_{t_{dk}}^{[d-1]}a_k \rightarrow w$ , hence

$$(\text{id}^{[d-1]} \times \theta_{t_{dk}}^{[d-1]})(b_k, a_k) = (b_k, \theta_{t_{dk}}^{[d-1]}a_k) \rightarrow (\pi_1^\bullet(v)x^{[d-1]}, w).$$

Since  $\text{id}^{[d-1]} \times \theta_{t_{dk}}^{[d-1]} = \tau_{(e, \dots, e, t_{dk})}^{[d]} \in \mathcal{F}_d$  and  $(b_k, a_k) \in \overline{\mathcal{F}_d(x^{[d]})}$ , we have

$$(5) \quad (\pi_1^\bullet(v)x^{[d-1]}, w) \in \overline{\mathcal{F}_d(x^{[d]})}.$$

Since, by assumption,  $(x^{[d-1]}, w)$  is  $\mathcal{F}_d$  minimal, there is some minimal idempotent  $u \in J(L)$  such that  $u(x^{[d-1]}, w) = (\pi_1^\bullet(u)x^{[d-1]}, \pi_2^\bullet(u)w) = (x^{[d-1]}, w)$ . Since  $u, v \in L$  are minimal idempotents in the same minimal left ideal  $L$ , we have  $uv = u$ . Thus  $u(\pi_1^\bullet(v)x^{[d-1]}, w) = (\pi_1^\bullet(u)\pi_1^\bullet(v)x^{[d-1]}, \pi_2^\bullet(u)w) = (\pi_1^\bullet(uv)x^{[d-1]}, w) = (\pi_1^\bullet(u)x^{[d-1]}, w) = (x^{[d-1]}, w)$ . By (5), we have  $(x^{[d-1]}, w) \in \overline{\mathcal{F}_d(x^{[d]})}$  and the proof of the lemma is completed.  $\square$

We are now ready to complete the proof of part (3) of the theorem. We assume by induction that this assertion holds for every  $1 \leq j \leq d-1$  and now, given  $x \in X$ , consider a minimal subflow  $Y$  of the flow  $(Q_x^{[d]}, \mathcal{F}_d)$ . With notations as in the previous lemma, we observe that  $Y_1 = \pi_1(Y)$  is a minimal subflow of the flow  $(Q_x^{[d-1]}, \mathcal{F}_{d-1})$  and therefore, by the induction hypothesis  $Y_1 = Y_x^{[d-1]} = \overline{\mathcal{F}_{d-1}x^{[d-1]}}$ . But then for some  $w \in Q^{[d-1]}$  we have  $(x^{[d-1]}, w) \in Y$  and, applying Lemma 1.3, we conclude that  $(x^{[d-1]}, w) \in Y_x^{[d]}$ . Thus  $Y = Y_x^{[d]}$  and the proof is complete.

4. Let  $2^{X^{[d]}}$  be the compact hyperspace consisting of the closed subsets of  $X^{[d]}$  equipped with the (compact metric) Vietoris topology. Let  $\Phi : X \rightarrow 2^{X^{[d]}}$  be the map  $x \mapsto Y_x^{[d]}$ . It is easy to check that this map is lower-semi-continuous (i.e.  $x_i \rightarrow x \Rightarrow \liminf \Phi(x_i) \supset \Phi(x)$ ). It follows then that the set of continuity points of  $\Phi$  is a dense  $G_\delta$  subset  $X_0 \subset X$  (see e.g. [2]). Since the set  $\mathcal{F}_d \Delta^{[d]}$  is dense in  $Q^{[d]}$ , it follows that at each point of  $X_0$  we must have  $Y_x^{[d]} = Q_x^{[d]}$ .  $\square$

## 2. $RP^{[d]}$ IS AN EQUIVALENCE RELATION

In this section we outline the Shao-Ye proof that  $RP^{[d]}$  is an equivalence relation. We assume that  $(X, T)$  is a minimal compact *metrizable*  $T$ -flow, where  $T$  is an abelian group. We fix a compatible metric  $\rho$  on  $X$ .

**2.1. Definition.** The *regionally proximal relation of order  $d$*  is the relation  $RP^{[d]} \subset X^{[d]} \times X^{[d]}$  defined by the following condition:  $(x, y) \in RP^{[d]}$  iff for every  $\delta > 0$  there is a pair  $x', y' \in X$  and  $(t_1, \dots, t_d) \in T^d$  such that:

$$\rho(x, x') < \delta, \quad \rho(y, y') < \delta \quad \text{and} \\ \rho^{[d]}(\tau_{(t_1, \dots, t_d)}^{[d]} x'_*, \tau_{(t_1, \dots, t_d)}^{[d]} y'_*) := \sup\{\rho(t_\epsilon x', t_\epsilon y') : \epsilon \subset \{1, \dots, d\}, \epsilon \neq \emptyset\} < \delta.$$

For  $d = 1$  this relation is the classical *regionally proximal relation*, see e.g. [1].

A convenient characterization of  $RP^{[d]}$  is provided by Host-Kra-Maass in [6, Lemma 3.3]. Among its implications one has the corollary that the relation  $RP^{[d]}$  is preserved under factors (Corollary 2.3 below).

**2.2. Lemma.** *Let  $(X, T)$  be a minimal flow. Let  $d \geq 1$  and  $x, y \in X$ . Then  $(x, y) \in RP^{[d]}$  if and only if there is some  $a_* \in X_*^{[d]}$  such that  $(x, a_*, y, a_*) \in Q^{[d+1]}$ .*

*Proof.* Suppose first that  $(x, y) \in RP^{[d]}$ . Fix an arbitrary point  $z \in X$ . Then, given  $\delta > 0$ , we first find a pair  $x', y' \in X$  and  $(t_1, \dots, t_d) \in T^d$  which satisfy the requirements in Definition 2.1, and then replace  $x'$  by  $sz$  and  $y'$  by  $tsz$ , with appropriate  $s, t \in T$ , so that  $\rho(x, sz) < \delta$ ,  $\rho(y, tsz) < \delta$  and

$$\rho^{[d]}(\tau_{(t_1, \dots, t_d)}^{[d]}(sz)_*, \tau_{(t_1, \dots, t_d)}^{[d]}(tsz)_*) < \delta.$$

Denoting  $a_{*\delta} = (sz)_*^{[d]}$  we have

$$\tau_{(t_1, \dots, t_d, t)}^{[d+1]}(sz, a_{*\delta}, sz, a_{*\delta}) = (\tau_{(t_1, \dots, t_d)}^{[d]}(sz, a_{*\delta}), \theta_t^{[d]} \tau_{(t_1, \dots, t_d)}^{[d]}(sz, a_{*\delta})) \in Q^{[d+1]}$$

Now, chose a convergent subsequence to get

$$\lim_{\delta \rightarrow 0} \tau_{(t_1, \dots, t_d, t)}^{[d+1]}(sz, a_{*\delta}, sz, a_{*\delta}) = (x, a_*, y, a_*) \in Q^{[d+1]}.$$

Conversely, assume that there is some  $a_* \in X_*^{[d]}$  such that  $(x, a_*, y, a_*) \in Q^{[d+1]}$ . Then, there exist sequences  $x_n \in X$  and  $F_n \in \mathcal{F}_{d+1}$  such that

$$F_n((x_n)^{[d+1]}) \rightarrow (x, a_*, y, a_*).$$

Now  $F_n$  has the form  $F_n = (\tau_n^{[d]}, \theta_{t_n}^{[d]} \tau_n^{[d]})$  with  $t_n \in T$  and  $\tau_n^{[d]} \in \mathcal{F}_d$ , so that  $x_n \rightarrow x$  and  $t_n x_n \rightarrow y$ , and it follows that  $(x, y) \in RP^{[d]}$ , as required.  $\square$

It follows directly from the definition that  $RP^{[d]}$  is a symmetric and  $T$ -invariant relation. It is also easy to see that it is closed. However, even for  $d = 1$  there are easy examples which show that, in general, it need not be an equivalence relation (not being transitive). The remarkable fact that when  $(X, T)$  is minimal, and  $T$  is abelian, the relation  $RP^{[1]}$  is an equivalence relation (and therefore coincides with the equicontinuous structure relation; i.e., the smallest closed invariant relation  $S \subset X \times X$  such that the quotient flow  $(X/S, T)$  is equicontinuous) is due to Ellis and Keynes [4] (see also [8]).

**2.3. Corollary.** *If  $\pi : (X, T) \rightarrow (Y, T)$  is a homomorphism of minimal  $T$ -flows then*

$$(\pi \times \pi)(RP^{[d]}(X)) \subset RP^{[d]}(Y).$$

Equipped with Theorem 1.1 we will now show that for every  $d \geq 1$  the relation  $RP^{[d]}$  is an equivalence relation. First we prove two more necessary and sufficient conditions on a pair  $(x, y) \in X \times X$  to belong to  $RP^{[d]}$ .

**2.4. Proposition.** *Let  $(X, T)$  be a minimal flow and  $d \geq 1$ . The following conditions are equivalent:*

1.  $(x, y) \in RP^{[d]}$ .
2.  $(x, y, y, \dots, y) = (x, y_*^{[d+1]}) \in Q^{[d+1]}$ .
3.  $(x, y, y, \dots, y) = (x, y_*^{[d+1]}) \in \overline{\mathcal{F}_{d+1}(x^{[d+1]})}$ .

*Proof.* (3)  $\Rightarrow$  (2) is obvious. The implication (2)  $\Rightarrow$  (1) follows from Lemma 2.2. Thus it suffices to show that (1)  $\Rightarrow$  (3). Let  $(x, y) \in RP^{[d]}$ , then by Lemma 2.2, there is some  $a \in X^{[d]}$  such that  $(x, a_*, y, a_*) \in Q^{[d+1]}$ . Observe that  $(y, a_*) \in Q^{[d]}$ . By Theorem 1.1.(3), there is a sequence  $\{F_k\} \subset \mathcal{F}_d$  such that  $F_k(y, a_*) \rightarrow y^{[d]}$ . Hence

$$(F_k \times F_k)(x, a_*, y, a_*) \rightarrow (x, y_*^{[d]}, y, y_*^{[d]}) = (x, y_*^{[d+1]}).$$

Since  $F_k \times F_k \in \mathcal{F}_{d+1}$  and  $(x, a_*, y, a_*) \in Q^{[d+1]}$ , we have that  $(x, y_*^{[d+1]}) \in Q^{[d+1]}$ . By Theorem 1.1.(2),  $y^{[d+1]}$  is  $\mathcal{F}_{d+1}$ -minimal. It follows that  $(x, y_*^{[d+1]})$  is also  $\mathcal{F}_{d+1}$ -minimal. Now  $(x, y_*^{[d+1]}) \in Q^{[d+1]}[x]$  is  $\mathcal{F}_{d+1}$ -minimal and by Theorem 1.1.(3),  $\overline{\mathcal{F}_{d+1}(x^{[d+1]})}$  is the unique  $\mathcal{F}_{d+1}$ -minimal subset in  $Q^{[d+1]}[x]$ . Hence we have that  $(x, y_*^{[d+1]}) \in \overline{\mathcal{F}_{d+1}(x^{[d+1]})}$ , and the proof is completed.  $\square$

As an easy consequence of Proposition 2.4 we now have the following theorem.

**2.5. Theorem.** *Let  $(X, T)$  be a minimal metric flow, where  $T$  an abelian group, and  $d \geq 1$ . Then  $RP^{[d]}$  is an equivalence relation.*



*Proof.* It suffices to show the transitivity, i.e. if  $(x, y), (y, z) \in RP^{[d]}$ , then  $(x, z) \in RP^{[d]}(X)$ . Since  $(x, y), (y, z) \in RP^{[d]}$ , by Proposition 2.4 we have

$$(y, x, x, \dots, x), (y, z, z, \dots, z) \in \overline{\mathcal{F}_{d+1}(y^{[d+1]})}.$$

By Theorem 1.1.(2)  $(\overline{\mathcal{F}_{d+1}(y^{[d+1]})}, \mathcal{F}_{d+1})$  is minimal, whence

$$(y, z, z, \dots, z) \in \overline{\mathcal{F}_{d+1}(y, x, x, \dots, x)}.$$

Finally, as  $\mathcal{F}_{d+1}$  acts as the identity on the  $\emptyset$ -coordinate, it follows that also

$$(x, z, z, \dots, z) \in \overline{\mathcal{F}_{d+1}(x^{[d+1]})}.$$

By Proposition 2.4 again,  $(x, z) \in RP^{[d]}$ . □

**2.6. Remark.** From Proposition 2.4 we deduce that in the definition of the regionally proximal relation of order  $d$  the point  $x'$  can be replaced by  $x$ . More precisely, a pair  $(x, y) \in X \times X$  is in  $RP^{[d]}$  if and only if for every  $\delta > 0$  there is a point  $y' \in X$  and  $(t_1, \dots, t_d) \in T^d$  such that:

$$\rho(y, y') < \delta \quad \text{and} \\ \rho^{[d]}(\tau_{(t_1, \dots, t_d)}^{[d]} x_*^{[d]}, \tau_{(t_1, \dots, t_d)}^{[d]} y'_*{}^{[d]}) := \sup\{\rho(t_\epsilon x, t_\epsilon y') : \epsilon \subset \{1, \dots, d\}, \epsilon \neq \emptyset\} < \delta.$$

Again for  $d = 1$  this is a well known result (see [10] and [8]).

Let us conclude with the following remark. It is not hard to see that the proximal relation  $P \subset X \times X$  is a subset of  $RP^d$  for each  $d \geq 1$  (see Proposition 3.1 in [6]). Thus for every  $d \geq 1$  the quotient flow  $X/PR^{[d]}$  is a minimal distal flow. Of course the main result of Host, Kra and Maass in this work [6] is the fact that, for  $T = \mathbb{Z}$ , this minimal distal factor flow is the maximal factor of  $(X, T)$  which is a *system of order  $d - 1$* ; i.e., a  $T$ -flow which is an inverse limit of  $(d - 1)$ -step minimal  $T$ -nilflows. In turn, the results in [6] are based on the profound analogous ergodic theoretical theorems obtained by Host and Kra in [5].

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